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Family of Local Solutions in a Missile-Aircraft Differential Game

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To date many studies of missile and aircraft differential games seem to have tried to find a unique solution, however, the authors of this study found there are many solutions of comparable values to the game. The authors called them "a family of local solutions." This paper shows a method to find the family for a missile and an aircraft differential game, which minimizes and maximizes the essential payoff of the problem, the miss distance, without employing any linearized approximation. For this purpose, the authors introduced the concept of energy maneuverability. The sets of optimal terminal surfaces of both vehicles that maximize their velocities are obtained first. Next, minimax solutions of the vehicles that start from these terminal surfaces, and minimize and maximize the miss distance, respectively, are calculated; then a local minimax family in this game is obtained. The proposed method can solve minimax problems under several vehicles' acceleration constraints. The results may be able to be incorporated into an actual missile guidance system as a knowledge base, and then the performance will be significantly improved.

Nomenclature

A	=	aircraft
a	=	normal acceleration
a_c	=	missile normal acceleration command signal
C_D, C_L	=	drag and lift coefficients, respectively
C_{D0}	=	zero-lift drag coefficient
D	=	drag
$\boldsymbol{f}$	=	state derivative vector
g	=	acceleration of gravity
h	=	altitude
H, H^*	=	Hamiltonian functions for minimum-maximum miss distance and maximum velocities problems, respectively
I_{sp}	=	specific impulse
J	=	performance index
k	=	induced drag of aircraft
k_1, k_2, k_m	=	drag coefficients of missile
L	=	lift
M	=	missile
m	=	mass
r	=	slant range between missile and aircraft
S	=	reference area
T	=	thrust
t	=	time
t_e	=	sustainer burning time
t_f	=	interception time
$\boldsymbol{u}$	=	control vector
v	=	velocity
x	=	horizontal coordinate
$\boldsymbol{x}$	=	state vector
z	=	normal distance between missile and aircraft
α, α_0	=	angle-of-attack and zero-lift angle, respectively
$\boldsymbol{\lambda}$, $\boldsymbol{\lambda}^*$	=	adjoint vectors for minimum-maximum miss distance and maximum velocities problems, respectively
ρ	=	air density
σ, γ	=	line-of-sight and flight path angles, respectively
τ	=	missile time constant
ϕ, ϕ^*	=	performance indexes for miss distance minimum-maximum and velocities maximum problems, respectively
Ω	=	terminal condition
$(\dot{\cdot})$	=	time derivative

Subscripts

0, f	=	initial and terminal values, respectively
a	=	aircraft
c	=	command signal
m	=	missile
max, min	=	maximum and minimum values, respectively

I. Introduction

PURSUIT-EVASION problems between a missile and an aircraft have attracted many researchers, and a large number of studies have already appeared. In many papers a missile employs PN (proportional navigation) or its improved versions [1–18]. Currently most existing missiles employ PN-based guidance, therefore this is reasonable. As the PN is an approximation of closed-loop optimal control against a nonmaneuvering target [19], therefore an aircraft optimal avoidance problem against a PN missile may be said to be a suboptimal missile vs optimal aircraft pursuit-evasion game. However, it is not a precise two-sided optimal control problem, it is natural to motivate researchers to solve a true two-sided optimal control problem: a differential game between two vehicles. The famous "homicidal chauffeur" problem, first solved by Isaacs [20], with the perfect solution later obtained by Merz [21] has attracted many researchers' interest, and has become a good reference for this purpose. That is, the missile-aircraft pursuit-evasion problem may be said to be a complicated version of the above problem. However, to apply to a missile and an aircraft and analytically solve the problem, the vehicle models have to be simplified. Many studies of differential games have also appeared up to now [2,3,5,11,22–33]. If researchers hope to employ practical complicated models, and select the exact miss distance (MD) as the payoff (note that, in some papers the

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problem is formulated as a regulator, and the payoff is not the exact MD, but that of the regulator), then numerical calculations are essential. A nonlinear optimal control problem is reduced to a two-point boundary value problem, and until the 1980s, many versions of nonlinear optimal control solvers have been developed [19,34,35]. Most of them employed Euler–Lagrange equations, and reduced the problems to two-point boundary value problems. The SCGRA (sequential conjugate gradient-restoration algorithm) [35] can optimize unknown parameters as well as control histories, which may be said to be one of the most excellent methods among them. They treat the problems as continuous systems, and numerically integrate the Euler–Lagrange equations. Up to the middle of the 1980s they were employed to apply to differential games. On the other hand, following the increase of computer speed, the development of NLP (nonlinear mathematical programming) was remarkable, and after the 1980s, these methods of discretizing time histories of control and state variables, and solving parameter optimization problems became popular [36–38]. In particular, excellent nonlinear optimal control solvers such as those employing DCNLP (direct collocation with nonlinear programming) were successfully employed. These algorithms were originally employed to solve one-sided optimal control problems, but later successfully modified and applied to solve two-sided optimal control problems [29–33]. Recently, except for one of the authors' papers [17], it seems existing papers all employed NLP to solve complicated differential games. One of the newest examples is shown in [33], which also provides information of the other references to the readers. However, because of the number of discretizations is still limited, the precision is not high using NLP. Therefore, for problems requiring high precision, the NLP solvers may have difficulties in application. To solve such problems, the authors have consistently been studying the application of the gradient algorithm. This paper shows one such study where the solution is obtained by the gradient method-based solver, and may be more difficult to solve using NLP.

This study shows a method to obtain a family of minimax solutions without employing any linearized approximation which minimizes and maximizes the essential payoff of the problem: the MD for differential games with realistic vehicle models. The authors' study [17] has shown the method to obtain a solution by starting from a solution of "PN missile vs an optimally avoiding aircraft," and iteratively improving the solution for "an optimally pursuing missile vs an optimally avoiding aircraft." In this paper, a method to obtain more general solutions for medium range cases is shown by introducing the concept of energy maneuverability. As a perfect information game is assumed, the aircraft feint movement which is an important factor for one-sided optimal control cases has no use as the missile knows the aircraft strategy. In the results, both vehicles try to maintain the most advantageous states until a very close point before the interception, where the aircraft takes the upward or downward maximum acceleration. The missile takes instant maximum or minimum acceleration commands corresponding to the aircraft movement. In the above situations the proposed method can solve minimax solutions under several vehicles' acceleration constraints. The details of the problem, the algorithm for the solution and the features of the result are shown in this paper.

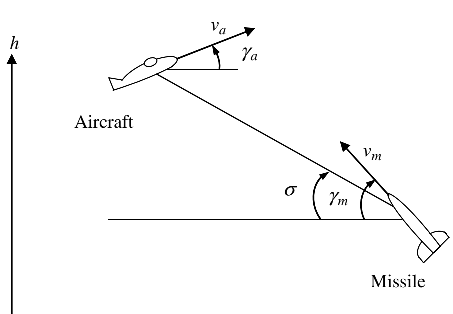


Fig. 1 Geometry and symbols.

II. Problem Formulation

Figure 1 shows the relative geometry of the pursuer and the evader and symbols. The authors' studies have shown that a vertical maneuver is more effective than a horizontal maneuver for the aircraft [4,6,7], therefore, the motions are constrained in a vertical plane. The reason is also explained in Sec. III. In the paper, the pursuer is a missile, and the evader is an aircraft. Both vehicles are modeled as point masses, and the equations of motion in a vertical plane are as follows:

A. Aircraft

$$\dot{v}_a = \frac{T_a \cos\alpha - D}{m_a} - g\sin\gamma_a \tag{1}$$

$$\dot{\gamma}_a = \frac{L + T_a \sin\alpha}{m_a v_a} - \frac{g}{v_a}\cos\gamma_a \tag{2}$$

$$\dot{x}_a = v_a \cos\gamma_a \tag{3}$$

$$\dot{h}_a = v_a \sin\gamma_a \tag{4}$$

$$L = \frac{1}{2}\rho_a v_a^2 S_a C_L, \qquad D = \frac{1}{2}\rho_a v_a^2 S_a C_D \tag{5}$$

$$C_L = C_{L\alpha}(\alpha - \alpha_0),\ C_D = C_{D0} + kC_L^2 \tag{6}$$

$$\rho_a = \rho_a(h_a), \qquad T_a = T_a(v_a, h_a) \tag{7}$$

A constraint is imposed on the value of the aircraft's normal acceleration a_a, which is the aircraft control variable:

$$a_a = L/m_a, \qquad |a_a| \le a_{a\max} \tag{8}$$

B. Missile

The missile's normal acceleration is approximated by a first-order lag to a normal acceleration command a_{mc}, which is the missile control variable

$$\dot{v}_m = \frac{T_m - D_m}{m_m} \tag{9}$$

$$\dot{a}_m = \frac{a_{\mathrm{mc}} - a_m}{\tau}, \qquad |a_{\mathrm{mc}}| \le a_{\mathrm{mc}\max} \tag{10}$$

$$\dot{\gamma}_m = \frac{a_m}{v_m} \tag{11}$$

$$\dot{x}_m = v_m \cos\gamma_m \tag{12}$$

$$\dot{h}_m = v_m \sin\gamma_m \tag{13}$$

where

$$D_m = k_1 v_m^2 + k_2\left(\frac{a_m}{v_m}\right)^2 \tag{14}$$

$$k_1 = \frac{1}{2}\rho_m S_m C_{D0m} \qquad k_2 = \frac{2k_m m_m^2}{\rho_m S_m} \tag{15}$$

$$T_m(t) = \begin{cases} T_m & \text{for } 0 < t \le t_e \\ 0 & \text{for } t_e < t \end{cases} \tag{16}$$

The purpose of this study is to obtain the optimal control histories of the aircraft and missile $a_a(t)$ and $a_{\mathrm{mc}}(t)$ that maximize and minimize the final MD, respectively. The interception time t_f is given from the following terminal condition

$$\Omega[\boldsymbol{x}(t_f), t_f] = [(x_a - x_m)(\dot{x}_a - \dot{x}_m) + (h_a - h_m)(\dot{h}_a - \dot{h}_m)]_{t=t_f} = 0 \tag{17}$$

The performance index ϕ is the MD: relative distance at t_f

$$\phi = [(x_a - x_m)^2 + (h_a - h_m)^2]_{t=t_f}^{\frac{1}{2}} \tag{18}$$

III. Some Features of Solutions in Former Studies

Some features of the optimal aircraft evasive maneuvers in a plane obtained in the authors' previous studies are as follows:

1) If the initial relative range is large, the aircraft takes the maximum normal acceleration to turn in the opposite direction, then flies away straight (type 1).

2) For a PN missile, and if the initial relative range is not large enough, the aircraft takes the maximum upward acceleration at first, then at an appropriate time switches to maximum downward acceleration (type 2).

3) In a perfect information DG where the initial relative range is medium, the aircraft takes an optimal trajectory (which is explained in a later section) at first, then at an appropriate time switches to its maximum upward or downward acceleration (type 3). Maneuvers type 2 and type 3 are called vertical-S and split-S, respectively.

In point 2, if the remaining time is large, but not enough to implement the type 1 maneuver, the aircraft may take a downward maneuver at first, then take the type 2 maneuver. This first motion of the aircraft is considered the maneuver to achieve the best relative geometry to the missile before implementing the type 2 maneuver. In the type 2 maneuver, the reason for the aircraft finally selecting a downward maneuver is explained in the following [4].

Suppose that there is no earth gravity, and consider a case where both vehicles are progressing in a head on geometry on the x axis. Also suppose that the initial normal accelerations of both vehicles are 0 and at a point in time the aircraft takes its maximum normal acceleration, while the missile instantly takes its maximum normal acceleration command, that is

$$a_a = a_m = 0 \quad \text{for } t < 0$$
$$a_a = a_{a\,\max}, \qquad a_{\mathrm{mc}} = a_{\mathrm{mc\,max}} \quad \text{for } t \ge 0 \tag{19}$$

then missile normal acceleration and its rate become

$$\dot{a}_m = (a_{\mathrm{mc\,max}} - a_m)/\tau \tag{20}$$

$$a_m = a_{\mathrm{mc\,max}}(1 - e^{-t/\tau}) \tag{21}$$

where τ is the missile time constant. The relative velocity of both vehicles $\dot{z}$ normal to x axis is

$$\dot{z} = \int_0^t (a_{a\,\max} - a_m)\,\mathrm{d}t = (a_{a\,\max} - a_{\mathrm{mc\,max}})t + \tau a_{\mathrm{mc\,max}}(1 - e^{-t/\tau}) \tag{22}$$

By integrating Eq. (22)

$$z = \frac{(a_{a\,\max} - a_{\mathrm{mc\,max}})}{2}t^2 + \tau a_{\mathrm{mc\,max}}[t - \tau(1 - e^{-t/\tau})] \tag{23}$$

$$\tau a_{\mathrm{mc\,max}}(1 - e^{-t_f/\tau}) = (a_{\mathrm{mc\,max}} - a_{a\,\max})t_f \tag{24}$$

By substituting Eq. (24) into Eq. (23) and denoting the z value at t_f as z_f, then

$$z_f = \frac{(a_{a\,\max} - a_{\mathrm{mc\,max}})}{2}t_f^2 + \tau t_f a_{a\,\max} \tag{25}$$

The maximum value of z_f occurs at $\mathrm{d}z_f/\mathrm{d}t_f = 0$. Denote the values of t_f and z_f as follows:

$$t_{f\,\max} = \frac{\tau a_{a\,\max}}{a_{\mathrm{mc\,max}} - a_{a\,\max}} \tag{26}$$

$$z_{f\,\max} = \frac{\tau^2 a_{a\,\max}^2}{2(a_{\mathrm{mc\,max}} - a_{a\,\max})} \tag{27}$$

The resultant MD is too pessimistic for an aircraft because of the instantaneous response of the missile. However, this equation explains that a downward maneuver has an advantage compared with an upward maneuver because the gravity acceleration g works on both vehicles, and in the right side of Eq. (27), in the denominator they cancel each other out, while in the numerator, the aircraft effective $a_{a\,\max}$ increases 1 g. The maximum normal acceleration of a modern fighter is at most 9 g, however, this 1 g increases the value to 10 g. The above result explains why a vertical maneuver is better for an aircraft than a horizontal maneuver.

Minimax solutions for type 1 and 2 are shown in the authors' former studies [4,6,7,10,13,21]. Obtaining the minimax solution for type 3 is the most difficult, and in this paper the algorithm to solve type 3 and the resulting solution are shown.

IV. Algorithm to Solve Type 3

Table 1 shows the initial conditions and some parameters of the vehicles. The scenario is, in a military engagement where the aircraft detected the missile at the condition shown in Table 1. The altitude is near the upper limit for the modeled aircraft to produce 9 g [39,40], and also with this initial altitude, the aircraft can safely conduct a

Table 1 Vehicle parameters

Aircraft (afterburner)			
$m_a = 7500$ kg	$h_{a0} = 4572$ m	$x_{a0} = 3000$ m	$S_a = 26$ m^2
$v_{a0} = 290.2$ m/s $(0.9M)$		$C_{L\alpha} = 3.689$/rad	
$C_{D0} = 0.0224$	$k = 0.260$	$a_{a\,\max} = 9$ g	
Engine: PWF-100	$T_{a\,\max} = T_{a\,\max}(v_a, h_a)$		
Missile (Sustainer phase)			
$m_{m0} = 173.6$ kg	$I_{\mathrm{SP}} = 250$ s	$T_m = 6000$ N	
$t_e = 8$ s	$S_m = 0.0324$ m^2	$h_{m0} = 4572$ m	$x_{m0} = 0$ m
$v_{m0} = 644.6$ m/s $(2.0M)$		$C_{L\alpha m} = 35.0$/rad	
$C_{D0m} = 0.90$	$k_m = 0.030$	$a_{\mathrm{mc\,max}} = 30$ g	
$\tau = 0.5$ s	$r_0 = 3000$ m	(head on)	

downward split-S, and recover his altitude. To obtain the minimax solution, the concept of the energy maneuverability, also called the specific energy: (potential energy + kinetic energy) per mass = $h + v^2/2$ is introduced [41,42]. The following assumption is employed: at the same altitude and with a larger energy maneuverability, and therefore a larger velocity, an aircraft can increase the MD, while a missile with a larger velocity can decrease the MD. This assumption seemed to be natural, but its proof is difficult. This problem will be discussed in Sec. VI. The authors' studies showed that, against a PN missile, an aircraft gradually increases upward acceleration until it reaches its maximum value, then at an appropriate time, the aircraft reverses his attitude and takes the maximum downward g. On the other hand, in a DG, as the missile does not know whether the aircraft will continue the maximum upward g, or reverse direction at any time, the optimal missile control should be the one that minimizes the MD in both cases. As a result, both the missile and aircraft try to keep maximum energy maneuverability until a very close point before the interception. Then, at an appropriate point from a velocity maximizing optimal control trajectory, the aircraft takes a last ditch maneuver (maximum normal upward or downward acceleration). The missile instantly responds to the aircraft maneuver and takes its maximum upward or downward acceleration command. The minimax DG solution branches into two trajectories depending on whether the aircraft takes maximum upward acceleration, or maximum downward acceleration.

In the proposed method, the steps to obtain this DG solution for a medium range case (type 3) are as follows:

1) To find the reachable region of the aircraft and efficiently search the solutions, the following preliminary calculations were conducted.

Let the aircraft take the maximum upward acceleration or minimum downward acceleration for the whole time, while the missile is guided by PN, and conduct simulations. In these cases the missile hits the aircraft at the time $t_f = 3.4$ s at the altitude $h_f = 4900$ m (called P1) and at the time $t_f = 3.3$ s at the altitude $h_f = 4350$ m (called P2) both without MD. The two obtained aircraft trajectories (called aircraft trajectories 1 and 2) are the upper bound and the lower bound of the aircraft optimal trajectories in the following study. Next, the missile velocity maximizing optimal trajectories that reach P1 and P2 (called missile trajectories 1 and 2) were calculated. The steepest ascent method [34] with fourth order Runge–Kutta–Gill integration with 0.05 s time step was employed to obtain these optimal controls and trajectories as well as the optimal controls and trajectories in the following study. In [34], Mayer-type problems which maximize the performance index $\phi[\boldsymbol{x}(t_f), t_f]$ with the terminal constraints $\psi[\boldsymbol{x}(t_f), t_f] = 0$ were solved. In the above calculations, the terminal values of $x(t_f)$ and $h(t_f)$ were fixed to those of P1 and P2, and the optimal control $a_m(t)$ which maximize $v_m(t_f)$ with optimal trajectories $[x_m(t), h_m(t)]$ were obtained. Next, the aircraft trajectory with its normal acceleration is always zero (called aircraft trajectory 3) and the missile trajectory with its normal acceleration command is always zero (called missile trajectory 3) were calculated, which intercept at the time $t_f = 3.3$ s at the altitude $h_f = 4519$ m (called P3).

Then, the following simulations were conducted. Starting from the aircraft trajectories 1–3 (and corresponding aircraft lateral accelerations), switch the aircraft normal acceleration as follows. For the aircraft trajectory 1, maximum upward to maximum downward, for the aircraft trajectory 2, maximum downward to maximum upward, and for the aircraft trajectory 3, two cases: zero acceleration to maximum upward acceleration, and zero acceleration to maximum downward acceleration. In these simulations, the acceleration switching time is selected from time $t = 0.1$ s through 3.3 s at time intervals of 0.1 s, where t is the elapsed time from the initial condition shown in Table 1. On the other hand, the missile starts from the missile trajectories 1–3, and takes the maximum upward or downward normal acceleration command corresponding to the direction of the aircraft normal acceleration at the same switching time as the aircraft. If the missile can equal the aircraft altitude by their interception time (CAP: closest approach point), it is clear that, an optimal control of the missile (not unique) exists which can attain zero MD, and the aircraft has no chance to escape from the missile. The above simulation results showed that, the time for the aircraft taking the last ditch maneuver t should be larger than 1.8 s.

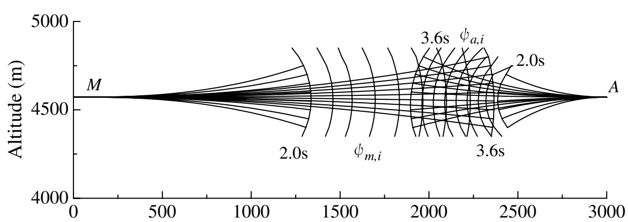


Fig. 2 Optimal trajectories and terminal surfaces of the aircraft and missile.

2) In reference to the above results, velocity maximizing optimal trajectory (with the accompanying optimal control) families of the aircraft and the missile were calculated for the final time t_f of 1.8 s through 3.6 s at intervals of 0.2 s, with the final altitude h_f specified from 4350 m through 4900 m at intervals of 50 m. In this study, the aircraft optimal controls are obtained by employing (1–8) with the following stopping condition, terminal constraint and the performance index

$$\Omega[\boldsymbol{x}(t_f), t_f] \equiv [h_a(t) - h_{af}]_{t=t_f} = 0, \qquad h_{af}\text{: specified} \tag{28}$$

$$\phi[\boldsymbol{x}(t_f), t_f] = v_a(t_f) \tag{29}$$

while, the missile optimal controls are obtained by employing (9–16) with the following stopping condition, terminal constraint and the performance index

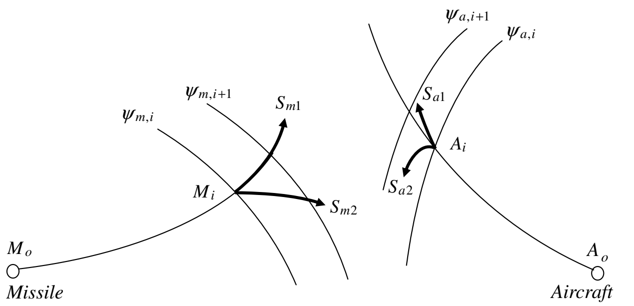
$$\Omega[\boldsymbol{x}(t_f), t_f] \equiv [h_m(t) - h_{mf}]_{t=t_f} = 0, \qquad h_{mf}\text{: specified} \tag{30}$$

$$\phi[\boldsymbol{x}(t_f), t_f] = v_m(t_f) \tag{31}$$

respectively. The optimal trajectories obtained in the above were interpolated by B-spline as functions of the time t, and the terminal surfaces (the surface of t_f is constant) were produced by connecting the above terminal points by using B-spline. Some of these terminal surfaces and optimal trajectories of vehicles are shown in Fig. 2, where the terminal surfaces are drawn from 2.0 through 3.6 s at every 0.2 s. The numbers of the terminal surfaces of the aircraft and the missile were increased to every 1ms also by employing B-spline interpolation, therefore the intercept region of the $x - h$ plane between the aircraft and the missile is covered by 1,801 terminal surfaces of the aircraft and the missile from the altitude h 4350–4900 m.

3) For convenience, define the time set $\{t_i\} = \{t_1, t_2, \ldots, t_{n-1}, t_n\}$, where $t_1 = 1.8$ s, $t_n = 3.6$ s, time interval $t_{i+1} - t_i = 0.001$ s, and $n = 1801$. For each t_i, $(i = 1 \sim n)$, calculate the aircraft trajectories starting from the point A_i on the aircraft terminal surface $\psi_{a,i}$, and the aircraft takes the maximum upward or downward acceleration, where A_i is selected on $\psi_{a,i}$ with every 2 m altitude increase from the lower bound through the upper bound of the aircraft altitude stated in point 1. Where these bounds do not exist, the minimum altitude of 4350 m and the maximum altitude of 4900 m were employed. The conceptual trajectories of the aircraft are shown as S_{a1} and S_{a2} in Fig. 3, respectively. Calculate the missile trajectories by starting from the point M_i on the missile terminal surface $\psi_{m,i}$ (which is the optimal location surface of the missile when the aircraft is on $\psi_{a,i}$), and the missile takes the maximum upward acceleration command against S_{a1} and maximum downward acceleration command against S_{a2}. If the missile can equal the aircraft altitude by their interception time against both S_{a1} and S_{a2}, it is clear that, an optimal control of the missile (not unique) exists which can attain zero MD.

4) Try to find the M_i on $\psi_{m,i}$ which produces nonzero MD, the process of which is explained in the following. The M_i search should be conducted in the region where the altitude of M_i is of similar



$\psi_{a,i}$: Aircraft optimal (velocity maximum) surface at t_i
$\psi_{m,i}$: Missile optimal (velocity maximum) surface at t_i
S_{a1} : Aircraft takes maximum upward acceleration at A_i
S_{a2} : Aircraft takes maximum downward acceleration at A_i
S_{m1} : Missile takes optimal control to minimize terminal miss against S_{a1} at M_i
S_{m2} : Missile takes optimal control to minimize terminal miss against S_{a2} at M_i

Fig. 3 The calculation of the minimax solution for general medium range cases.

altitude to A_i. For a small value of i, for arbitrary A_i on $\psi_{a,i}$, some M_is on $\psi_{m,i}$ will always be found where both MD against S_{a1} and S_{a2} are 0. In these cases, the aircraft does not have a chance to escape from the missile. However, by increasing the t_i value, an i, and A_i will be found where both missile maneuvers with maximum upward and downward acceleration commands from any M_i cannot reduce MDs against both S_{a1} and S_{a2} simultaneously to zero. Then select the M_i which produces the same value of MDs against both S_{a1} and S_{a2}, which is implemented by slightly changing the altitude of the M_i.

Work out the process on the arbitrary point of $\psi_{a,i}$ for all $\{t_1, t_2, \ldots, t_{n-1}, t_n\}$ for every 2 m interval of the altitude. Then the contours of the values of the minimax MD can be drawn. In these search parameters, the obtained aircraft control and corresponding missile control which produce the largest minimax value of MD may be called the solution to this DG. Figure 4 shows the final part of vehicle trajectories where the missile cannot capture the aircraft for both upward and downward maneuvers. Figure 5 shows the concept of a minimax solution. Figure 6 shows the locations the aircraft takes with its maximum upward or downward normal acceleration and simultaneously the missile takes with its maximum corresponding upward or downward normal acceleration commands resulting in the MD of 1.2–2.4 m. The same symbols correspond to the locations of both vehicles when the aircraft starts his last ditch maneuvers. The altitude differences of both vehicles at the start points are small, therefore the locations correspond to each other from low altitude to high altitude. This figure also shows that, as the relative distance decreases, the MD first increases from 1.2 m to more than 2.4 m, and then decreases to 1.2 m. The closing velocity is about 1000 m/s, therefore the interception time between two vehicles can also be

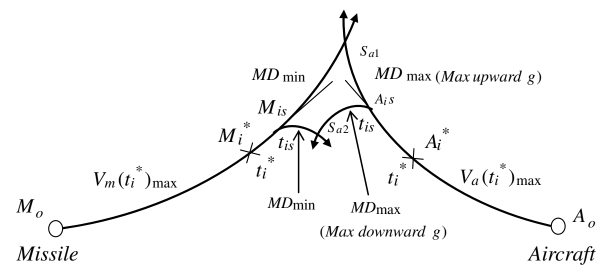


Fig. 5 The concept of a minimax solution.

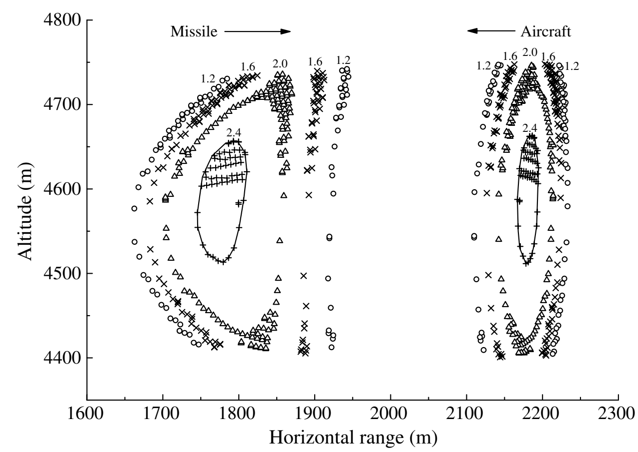


Fig. 6 The positions of vehicles corresponding to the MD of 1.2 m–2.4 m.

estimated. As the minimax value does not change uniformly depending on the increase or decrease of their altitudes and relative distances, they do not form a smooth curved plane, therefore the locations of the symbols of the same MD which appear in Fig. 6 are irregular. In this figure, the regions enclosed by bold lines show MDs mostly larger than 2.4 m, and the largest MDs are about 2.44 m (note that there are many locations which produce this MD). This value is far smaller than the values of more than 10 m obtained by the aircraft one-sided optimal control in the authors' previous studies [4,6,7]. Contrary to the studies up to now [3,17,24,25,27,32], (although it was known that the game value is unique but the optimal strategy is not necessarily unique) it turned out that the value of the game, MD in the global solution, is almost the same as other local optimum solutions, therefore trying to find the unique solution of this game problem is not practical. The locations of the 2.44 m MD are distributed (not unique) inside the lower half of the 2.4 m ellipse where a more than 2.4 m's MD is assured. That is, the aircraft may start his last ditch maneuver at any point from that region, which can

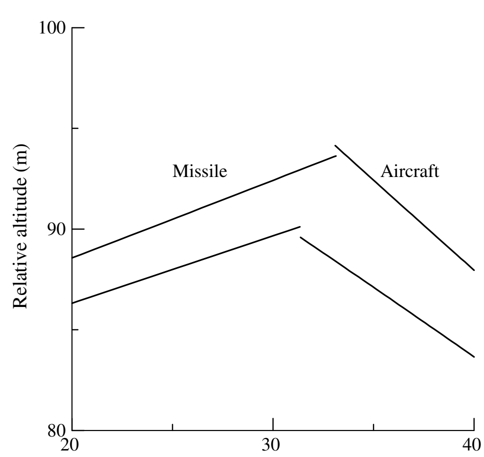


Fig. 4 The enlarged figure near interception points.

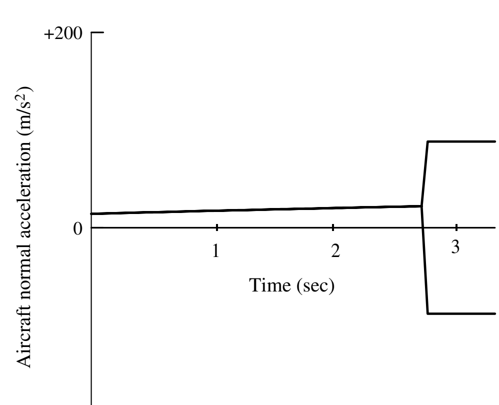


Fig. 7 A typical optimal aircraft control against a DG missile in medium range.

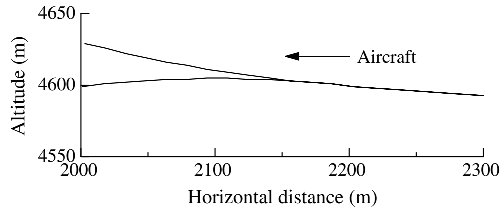


Fig. 8 A final part of optimal aircraft trajectories against a DG missile in medium range.

expect almost the same MD as the globally optimal solution. The set of these local solutions is named a "family of local minimax solutions." Figures 7 and 8 show typical aircraft control histories and parts of the trajectories.

V. Features of Solutions and Some Comments

Throughout this study, nonlinear optimal control problems were solved by the steepest ascent code, which has been developed by the authors. The principle of the algorithm is explained in [25,34]. Although the algorithm is an old one, several techniques such as searching for the solution automatically, avoiding local optima, and accelerating convergence are introduced in the code. The features of the optimal strategies of the aircraft and the missile are briefly summarized as follows. As a perfect information game is assumed, the aircraft feint movement (which is an important factor of type 2) has no use as the missile knows the aircraft strategy. However, the aircraft has the initiative to take an action at any time while the missile can not take an action until the aircraft takes an active strategy. As a result, the aircraft tries to maximize his velocity and obtain the most advantageous location in reference to the missile before his last ditch maneuver. As for the case of this paper, they are more concretely explained as follows. Starting from the initial condition shown in Table 1, the aircraft maintains its level flight (with 1 g's upward acceleration for the compensation of gravity) and increases his velocity. When the relative distance decreased to 410 m, the aircraft takes his last ditch maneuver (the location is illustrated as A_{is} in Fig. 5), where if the altitude of the missile is lower than the aircraft by more than 0.82 m , then the aircraft takes the maximum upward acceleration, while if the missile is higher than the altitude, then the aircraft takes the maximum downward acceleration. The relative distance of the location of A_{is} is not necessary precise. It can be relaxed to a value between 370 and 450 m. However, the difference of the altitude 0.82 m (corresponding to the relative distance of 370–450 m, changes by 0.67–0.99 m) is important, as the value is directly related to the value of MD achieved in this game. To maximize the aircraft velocity, as the induced drag is proportional to C_L^2 in Eq. (6), the aircraft takes almost a constant vertical acceleration, and in this case 1 g is best (at least one of the best ones as there are many local optimal solutions which produce the same value of MD).

As for the missile, it also takes a near constant 1 g value of upward acceleration, but it gradually decreases its altitude at first. Then, as the relative distance decreases, it gradually recovers its altitude, and at the location where the relative distance is 410 m, and the attitude is 0.82 m below than the aircraft where the aircraft takes his last ditch maneuver, the missile takes instant maximum or minimum acceleration commands corresponding to the direction of the aircraft acceleration, therefore the final situation always becomes like those shown in Fig. 4. In the optimal aircraft maneuvers against PN, from at a few seconds before interception, the aircraft always employs his maximum normal accelerations and the maneuvers become bang–bang types. With the aircraft parameters shown in Tables 1 and 2, and if the aircraft takes the normal acceleration of 9 g, the velocity deceleration is $-8.92\ \mathrm{m/s^2}$, while with 1 g level flight, the velocity increases with $+8.83\ \mathrm{m/s^2}$. In the optimal aircraft maneuvers against DG based missiles, to increase his velocity the aircraft maintains its normal small acceleration until the critical point of about 0.4 s before the interception.

One important result of this study is that a PN missile tends to take a higher altitude than an aircraft, while an intelligent missile, considering the result of the DG in this study, should maintain maximum energy maneuverability with a slightly lower altitude than the aircraft. A very short time before the interception, the missile can switch to PN for real time guidance considering the case of the aircraft takes a bang–bang type maneuver. As the experience of the authors' studies [4,6,7,43] has shown, the aircraft feint maneuvers against PN missiles require a relative distance of larger than about 800 m, therefore, within the range, the missile can switch to PN. The increase of the MD caused by this guidance law switching is not so large. In the case of this paper, the aircraft took rather straight trajectories. The reason is considered to be that the aircraft maximum acceleration is limited. On the other hand, if the aircraft maximum angle-of-attack is limited, the aircraft will try to take a lower altitude because it can produce a larger normal acceleration at a lower altitude. As for this point, a little more will be stated later.

The purpose of this paper is to show a method to solve this complicated differential game, and the parametric study in relation to the vehicles' initial geometries is beyond the scope of this paper. However for the interested readers, some comments are stated. As for the effect of the missile time constant τ and the aircraft maximum normal acceleration $a_{a\,\max}$, Eq. (27) is useful. Although an appropriate adjusting coefficient has to be multiplied to the right side of Eq. (27), MD is generally proportional to the square of τ and the square of $a_{a\,\max}$. As for the effect of the altitude, the aircraft velocity deceleration with 9 g's normal acceleration is $-5.03\ \mathrm{m/s^2}$ at 3048 m (10,000 ft), $-8.92\ \mathrm{m/s^2}$ at 4072 m (15,000 ft), and $-15.99\ \mathrm{m/s^2}$ at 6096 m (20,000 ft), respectively. In the optimal aircraft maneuver against DG based missile, the aircraft employs the maximum g very short time, therefore the effect of the altitude is not clearly appeared. However, in the optimal evasive maneuvers against PN missiles, the aircraft employs the maximum g a longer time, and at a high altitude, the aircraft velocity decreases very quickly, which decreases the MD. In particular, if the aircraft maximum angle-of-attack is limited for

Table 2 Aircraft thrust in relation to altitude and velocity (in N)

	$H(m)$								
V (m/s)	0.	1500.	3000.	4500.	6000.	7500.	9000.	10500.	12000.
0.	79125.	66047.	54782.	45135.	36910.	29951.	24095.	19207.	14664.
30.	81695.	68885.	57731.	48069.	39744.	32617.	26548.	21416.	16481.
60.	84609.	72018.	60935.	51224.	42764.	35436.	29129.	23735.	18381.
90.	89642.	76783.	65357.	55251.	46369.	38612.	31884.	26096.	20296.
120.	95057.	81784.	69934.	59407.	50106.	41938.	34818.	28652.	22374.
150.	100875.	87214.	74919.	63903.	54104.	45442.	37843.	31230.	24460.
180.	107582.	33936.	40994.	49103.	58315.	68701.	80333.	93263.	26660.
210.	114305.	44488.	53014.	62676.	73555.	85732.	99278.	37033.	29227.
240.	119912.	104572.	90718.	78237.	67049.	57068.	48210.	40464.	32110.
270.	123968.	109120.	95637.	83308.	72098.	61954.	52832.	44720.	35653.
300.	129616.	115166.	101699.	89269.	77865.	67449.	57985.	49441.	39544.
330.	134704.	120816.	107702.	95345.	83795.	73076.	63194.	54167.	43435.

preventing the aircraft from stall, the MD is almost inversely proportional to the altitude [7].

In this study, 240 optimal controls (from 1.8 through 3.6 s at every 0.2 s interval, and from 4350 through 4900 m at every 50 m interval) which maximize the velocities of the missile and the aircraft were initially calculated, and the average number of time steps was 54 (from 36 to 72). By the steepest ascent code, these were not so difficult. Next, reachable surfaces at 1ms intervals were calculated by using B-spline interpolations. Then, following to the algorithm stated in Sec. IV, minimax solutions were obtained. Recently almost all trajectory optimization and DG problems have been solved by using NLP. The NLP solvers discretize time histories of control and state variables, and treat them as unknown parameters and solve the nonlinear parameter optimization problems. The calculation by NLP with the same precision is very time consuming [37] and therefore may be more difficult. That is, classic gradient method-based solvers seem to be potent in solving DG problems.

VI. Discussion on the Optimality of the Solutions

In this section, the optimality of the obtained solutions is discussed. The state variables and their derivatives, and control variables in Eqs. (1–4) and (9–13) are expressed as follows:

$$\boldsymbol{x} = (v_a, \gamma_a, x_a, h_a, v_m, a_m, \gamma_m, x_m, h_m)^T \tag{32}$$

$$\boldsymbol{f} = \dot{\boldsymbol{x}} = (\dot{v}_a, \dot{\gamma}_a, \dot{x}_a, \dot{h}_a, \dot{v}_m, \dot{a}_m, \dot{\gamma}_m, \dot{x}_m, \dot{h}_m)^T \tag{33}$$

$$\boldsymbol{u} = (a_a(t), a_{\mathrm{mc}}(t))^T \tag{34}$$

The Hamiltonian function H and adjoint equations are given as follows: (see, e.g., [19])

$$H = \boldsymbol{\lambda}^T \boldsymbol{f} \tag{35}$$

$$\dot{\boldsymbol{\lambda}}^T = -\boldsymbol{\lambda}^T \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \tag{36}$$

$$\boldsymbol{\lambda}(t_f) = -\left(\frac{\partial \phi}{\partial \boldsymbol{x}}\right)^T_{t=t_f} \tag{37}$$

All local solutions of this study satisfy the next conditions

$$\begin{aligned} &|a_a(t)| = a_{a\max}, \quad |a_{\mathrm{mc}}(t)| = a_{\mathrm{mc}\max} \quad a_a(t) = \arg\max\{H\},\\ &a_{\mathrm{mc}}(t) = \arg\min\{H\} \quad \text{for } t_s \le t \le t_f \end{aligned} \tag{38}$$

where t_s is the time when the minimax trajectories of the vehicles branch into two curves. In Eq. (37) t_f and ϕ are determined from Eqs. (17) and (18). In Sec. IV, $a_a(t)$ and $a_{\mathrm{mc}}(t)$ are determined to maximize and minimize the performance index Eq. (18), respectively, therefore it is clear that the above conditions are satisfied.

All local solutions of this study also satisfy the next conditions

$$\begin{aligned} &\frac{\partial H^*}{\partial a_{\mathrm{mc}}} = 0, \quad \frac{\partial H^*}{\partial a_a} = 0 \quad a_a(t) = \arg\max\{H\},\\ &a_{\mathrm{mc}}(t) = \arg\min\{H\} \quad \text{for } 0 \le t \le t_s \end{aligned} \tag{39}$$

where

$$H^* = \boldsymbol{\lambda}^{*T} \boldsymbol{f} \tag{40}$$

$$\dot{\boldsymbol{\lambda}}^{*T} = -\boldsymbol{\lambda}^{*T} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \tag{41}$$

$$\boldsymbol{\lambda}^*(t_f) = -\left(\frac{\partial \phi^*}{\partial \boldsymbol{x}}\right)^T_{t=t_s} \tag{42}$$

$$\phi^* = (v_a - v_m)_{t=t_s} \tag{43}$$

Equation (43) means the aircraft tries to maximize ϕ^*, while the missile tries to minimize it. In other words, both vehicles try to maximize their velocities at t_s. The algorithm of the steepest ascent method [34] assures that Eq. (39) is satisfied. Unfortunately, the above Hamiltonian is that of maximizing both vehicles' velocities, but not that of minimizing and maximizing MD, therefore it cannot be the proof of the necessary condition of this DG. The concept of the energy maneuverability is widely introduced in air combat tactics [40,41] and missile guidance [39]. The equivalent time constants of the vehicles include the time constants of the airframes and the actuators the values of which decrease as the velocity of the vehicles increase. That is, with a smaller value of the missile time constant, the missile can reduce the MD, and with a smaller value of the aircraft time constant, the aircraft can increase the MD. Therefore, if the time constants of both vehicles are expressed as functions of their velocities, the assumption employed in Sec. IV that "with a larger velocity an aircraft can increase the MD, while a missile with a larger velocity can decrease the MD" can be easily proved. However, in the vehicle models in Sec. III, the missile time constant is assumed to be a constant and the aircraft time constant is neglected. With larger velocities, both vehicles can produce larger normal accelerations, which can decrease and increase the MD, respectively; however, the accelerations are limited in the vehicle models in Sec. III. The vehicle models in Sec. III are employed because almost all existing papers as well as the authors' employed the vehicle models with fixed values of time constants and normal acceleration limits of both vehicles.

However, the assumption has been well proved in existing studies, e.g., [7,40] where the same vehicle models in this paper are employed and the relation between the missile velocity versus MD are studied at several altitudes, and the decrease of the MD with the increase of the missile velocity is clearly shown. The reason for the results of the studies are considered to be, the deceleration of the missile caused by the drag gives the aircraft an additional time margin to the original time constant, and this time margin increases as the missile velocity decreases. The similar reason will be applied to the aircraft, and existing studies (e.g., [6,7,10,12]) have shown that, in aircraft optimal evasive maneuvers against PN missiles, the aircraft always use their maximum thrusts. These results may prove the fact that "with a larger velocity an aircraft can increase the MD."

The saddle point condition (Nash Equilibrium) of this DG is as follows:

$$\begin{aligned} &J[a^0_{\mathrm{mc}}(t), a_a(t)] < J[a^0_{\mathrm{mc}}(t), a^0_a(t)] < J[a_{\mathrm{mc}}(t), a^0_a(t)]\\ &\text{for } 0 \le t \le t_f \end{aligned} \tag{44}$$

where $a^0_{\mathrm{mc}}(t)$ and $a^0_a(t)$ are optimal controls of $a_{\mathrm{mc}}(t)$ and $a_a(t)$, respectively. Under the assumption employed in Sec. IV, the above characteristics are assured by the proposed algorithm. It is also verified by selecting arbitrary differential changes from the optimal control of one vehicle and calculating corresponding one-sided optimal control of the opponent vehicle and conducting simulations.

VII. Conclusions

A missile-aircraft pursuit-evasion differential game with very accurate complex vehicle models was solved by employing a steepest ascent method. To solve this difficult problem, the concept of "energy maneuverability" was introduced. The obtained value of the miss distance is 2.44 m, which is far smaller than the values of more than 10 m obtained by the aircraft one-sided optimal control in the authors' studies. Contrary to the results of previous studies, it turned out that there are many solutions of comparable values of the game; therefore, trying to find the globally optimal solution is not practical. That is, the aircraft may employ many local optimal solutions, which can expect almost the same MD as the globally optimal solution. A set of these solutions was called "a family of local minimax solutions." The process for obtaining the set was explained in detail. Although recently almost all differential game problems have been solved by using NLP solvers, classic gradient method-based solvers seem to be potent in solving differential games. The results of this study may be able to be incorporated into an actual missile guidance system as a knowledge base, then the performance will be significantly improved.

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